

OPTIMAL INJECTIVE STABILITY FOR THE SYMPLECTIC $K_1\mathrm{Sp}$ GROUP

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ABSTRACT. If R is a commutative ring, I an ideal of R and $\mathbf{v}, \mathbf{w} \in \mathrm{Um}_{2n}(R, I)$ then we show that \mathbf{v}, \mathbf{w} are in the same orbit of elementary action if and only if they are in that of elementary symplectic action extending a result in [4]. We also show that if A is a non-singular affine algebra of dimension d over an algebraically closed field k such that $d!A = A$, $d \equiv 2 \pmod{4}$ and I an ideal of A , then $\mathrm{Um}_d(A, I) = e_1\mathrm{Sp}_d(A, I)$. As a consequence it is proved that if A is a non-singular affine algebra of dimension d over an algebraically closed field k such that $(d+1)!A = A$, $d \equiv 1 \pmod{4}$ and I a principal ideal then $\mathrm{Sp}_{d-1}(A, I) \cap \mathrm{ESp}_{d+1}(A, I) = \mathrm{ESp}_{d-1}(A, I)$. We give an example to show that the above result does not hold true for an affine algebra over a C_2 field and also show by an example that the above stability estimate is optimal.

1. INTRODUCTION

We shall confine ourselves to working over a commutative ring R with 1, which is generally an affine algebra over an algebraically closed field k .

Historically, Bass–Milnor–Serre in [3] observed that $K_1(R) = \lim GL_n(R)/E_n(R)$ is a finite limit, attained when $n \geq d+3$, where d is the Krull dimension of R . They conjectured that the correct estimate should be $\max\{3, d+2\}$, which was established by L.N. Vaserstein in [20]. Vaserstein went on to show that a similar phenomenon occurs for the symplectic and orthogonal K_1 functors; and established estimates $\geq 2d+4$ for them.

A.A. Suslin started improving the estimates of J-P Serre and Hyman Bass for K_0 , by working over affine algebras over a field of cohomological dimension at most one (under certain divisibility conditions on the field; which we shall blur over in the introduction). This led Ravi A. Rao and W. van der Kallen in [12] to question whether a similar phenomenon would occur for injective K_1 -stabilization; which they established as $\max\{3, d+1\}$. Later in [5] it was further improved to $\max\{3, d\}$ under certain divisibility conditions on the field. In this article we prove similar estimate for relative K_1 (see Theorem 4.8).

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Their methods were successfully used in the symplectic group in the work of R. Basu–R.A. Rao in [1] to get the bound $\geq d + 1$ for the symplectic group. Along with S. Jose in [10] they also showed that the injective stability bound does not improve for the orthogonal K_1 even over nice affine algebras.

The Basu–Rao results were reproved (and slightly improved) by Chattopadhyay–Rao in [4]. Later Basu–Chattopadhyay–Rao in [2] succeeded in getting the injective estimate $\max\{4, d\}$, for smooth affine algebras over a field of cohomological dimension ≤ 1 , when d is divisible by 4.

There was a further improvement in the stability estimate for K_0 in the recent work of Jean Fasel–Ravi A. Rao–Richard G. Swan in [5], over a smooth affine algebra over an algebraically closed field k . They also showed that the injective stability estimate fell to $\max\{3, d\}$.

This led us to further investigate whether the injective stability bound for the symplectic group also fell similarly. In this article we first show the following result of [4] extending it to the case when $n = 2$ and removing the condition $2R = R$.

Theorem 1.1. *Let R be a commutative ring, I an ideal of R and $\mathbf{v} \in \text{Um}_{2n}(R, I)$. Then $\mathbf{v} \text{E}_{2n}(R, I) = \mathbf{v} \text{ESp}_{2n}(R, I)$.*

The above result is used to show the following.

Theorem 1.2. *Let A be a non singular affine algebra of dimension d over an algebraically closed field k such that $d \equiv 2 \pmod{4}$, $\frac{1}{d!} \in k$ and I an ideal of A . Let $\mathbf{v} \in \text{Um}_d(A, I)$. Then \mathbf{v} can be completed to a symplectic matrix which is congruent to identity modulo I .*

Then using Theorem 1.2 we show that the injective stability for symplectic K_1 falls to $\max\{4, d - 1\}$, when $d \equiv 1 \pmod{4}$ and $(d + 1)!A = A$. Precisely we prove the following.

Theorem 1.3. *Let A be a nonsingular affine algebra of dimension d over an algebraically closed field k and $I = (a)$ a principal ideal. Assume $(d + 1)! \in k^* = k \setminus \{0\}$, $d \equiv 1 \pmod{4}$. Then*

$$\text{Sp}_{d-1}(A, I) \cap \text{ESp}_{d+1}(A, I) = \text{ESp}_{d-1}(A, I).$$

We also show by an examples that one cannot expect the injective stability bounds to fall further for smooth affine algebras over an algebraically closed field. Just as the positive solutions of the cancellation problem for A^{d-1} leads to positive solution; the negative solution due to Mohan Kumar in [9] leads to the examples that the injective stability estimates will not fall further. These examples are also used by us to show that the injective stability estimates do not even fall to $d - 1$, as above, when we work with affine algebras over a field of cohomological dimension two. It remains to see if the situation improves over a field of cohomological dimension at most one.

Throughout this article R will denote a commutative ring with $1 \neq 0$ and I an ideal of R unless otherwise specified.

2. PRELIMINARIES

A row $\mathbf{v} = (v_1, v_2, \dots, v_n) \in R^n$ is said to be unimodular if there exists another row $\mathbf{w} = (w_1, w_2, \dots, w_n) \in R^n$ such that $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n v_i w_i = 1$. We shall denote the set of all unimodular rows in R^n by $\text{Um}_n(R)$. Let e_i denote the i -th row of the identity matrix I_n of size n . Then $\text{Um}_n(R, I)$ will denote the set of all unimodular rows of length n which are congruent to e_1 modulo I . It can be shown that for any $\mathbf{v} \in \text{Um}_n(R, I)$ there exists $\mathbf{w} \in \text{Um}_n(R, I)$ such that $\langle \mathbf{v}, \mathbf{w} \rangle = 1$. Any subgroup G of $\text{GL}_n(R, I) = \{\alpha \in \text{GL}_n(R) : \alpha \equiv I_n \pmod{I}\}$ acts on $\text{Um}_n(R, I)$ where I_n denotes the identity matrix. Let $\mathbf{v}, \mathbf{w} \in \text{Um}_n(R, I)$, we write $\mathbf{v} \sim_G \mathbf{w}$ if $\mathbf{v} = \mathbf{w}g$ for some $g \in G$.

Definition 2.1. (Elementary group $E_n(R)$, Relative elementary group $E_n(R, I)$) *Given $\lambda \in R$, for $i \neq j$, let $E_{ij}(\lambda) = I_n + \lambda e_{ij}$ where $e_{ij} \in M_n(R)$ is the matrix whose only non-zero entry is 1 at the (i, j) -th position. Such $E_{ij}(\lambda)$'s are called elementary matrices. The subgroup of $\text{GL}_n(R)$ generated by $E_{ij}(\lambda), i \neq j, \lambda \in R$ is called the elementary subgroup of $\text{GL}_n(R)$ and is denoted by $E_n(R)$. Similarly we define $E_n(I)$ for any ideal I of R . We define $E_n(R, I)$ to be the normal closure of $E_n(I)$ in $E_n(R)$. It is the smallest normal subgroup of $E_n(R)$ containing the element $E_{21}(x), x \in I$.*

Definition 2.2. (Symplectic group $\text{Sp}_{2n}(R)$, Relative symplectic group $\text{Sp}_{2n}(R, I)$) *We define the symplectic group $\text{Sp}_{2n}(R)$ to be the isotropy group of the standard symplectic form $\psi_n = \sum_{i=1}^n e_{2i-1} 2i - \sum_{i=1}^n e_{2i} 2i-1$. In other words $\text{Sp}_{2n}(R) = \{\alpha \in \text{GL}_{2n}(R) : \alpha^t \psi_n \alpha = \psi_n\}$. We define the relative symplectic group as $\text{Sp}_{2n}(R, I) = \{\alpha \in \text{Sp}_{2n}(R) : \alpha \equiv I_{2n} \pmod{I}\}$. Note that if $\alpha \in \text{Sp}_{2n}(R, I)$, then its transpose $\alpha^t \in \text{Sp}_{2n}(R, I)$.*

Let $\sigma \in S_{2n}$ denote the permutation of the natural numbers given by $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i; i = 1, 2, \dots, n$.

Definition 2.3. ($\text{ESp}_{2n}(R)$, $\text{ESp}_{2n}(R, I)$) *We define for $z \in R, 1 \leq i \neq j \leq 2n$,*

$$se_{ij}(z) = \begin{cases} 1_{2n} + ze_{ij} & \text{if } i = \sigma(j); \\ 1_{2n} + ze_{ij} - (-1)^{i+j} ze_{\sigma(j)\sigma(i)} & \text{if } i \neq \sigma(j) \text{ and } i < j. \end{cases}$$

It is easy to see that all these generators belong to $\text{Sp}_{2n}(R)$. We call them elementary symplectic matrices over R , and the subgroup of $\text{Sp}_{2n}(R)$ generated by them is called the elementary symplectic group $\text{ESp}_{2n}(R)$. Similarly the subgroup generated by $se_{ij}(z), z \in I$ is denoted by $\text{ESp}_{2n}(I)$. The group $\text{ESp}_{2n}(R, I)$ is defined to be the smallest normal subgroup of $\text{ESp}_{2n}(R)$ containing $\text{ESp}_{2n}(I)$.

The following is easy.

Lemma 2.4. *If the first row of a symplectic matrix $M \in \text{Sp}_{2n}(R, I)$ equals e_1 , then its second column is equal to e_2^t and vice versa.*

Definition 2.5. (The excision ring) *If I is an ideal of R , one constructs the ring $\mathbb{Z} \oplus I$ with multiplication defined by*

$$(n, i)(m, j) = (nm, nj + mi + ij)$$

for $m, n \in \mathbb{Z}, i, j \in I$. If the dimension of the ring is $d \geq 1$, then the maximal spectrum of $\mathbb{Z} \oplus I$ is the union of finitely many subspaces of dimension at most d ([19], 3.19).

There is a natural homomorphism $\mathfrak{f} : \mathbb{Z} \oplus I \rightarrow R$ given by $(m, i) \mapsto m + i \in R$. Let $\mathbf{v} = (1 + i_1, i_2, \dots, i_n) \in \text{Um}_n(R, I)$ where i_j 's are in I . Then we shall say $\tilde{\mathbf{v}} = (\tilde{1} + \tilde{i}_1, \tilde{i}_2, \dots, \tilde{i}_n) \in \text{Um}_n(\mathbb{Z} \oplus I, 0 \oplus I)$ for $\tilde{1} = (1, 0), \tilde{i}_j = (0, i_j)$ to be a lift of \mathbf{v} . Clearly \mathfrak{f} sends $\tilde{\mathbf{v}}$ to \mathbf{v} .

We shall define $\text{MSE}_n(R) = \frac{\text{Um}_n(R)}{\text{E}_n(R)}$ and likewise $\text{MSE}_n(R, I) = \frac{\text{Um}_n(R, I)}{\text{E}_n(R, I)}$. We recall the Excision theorem (see [19], Theorem 3.21).

Theorem 2.6. (Excision theorem) *Let $n \geq 3$ be an integer and I be an ideal of a commutative ring R . Then the natural maps $F : \text{MSE}_n(\mathbb{Z} \oplus I, 0 \oplus I) \rightarrow \text{MSE}_n(R, I)$ defined by $[(a_i)] \mapsto [(\mathfrak{f}(a_i))]$ and $G : \text{MSE}_n(\mathbb{Z} \oplus I, 0 \oplus I) \rightarrow \text{MSE}_n(\mathbb{Z} \oplus I)$ defined by $[(a_i)] \mapsto [(a_i)]$ are bijections.*

Definition 2.7. *We shall say a ring homomorphism $\phi : B \twoheadrightarrow D$ has a section if there exists a ring homomorphism $\gamma : D \hookrightarrow B$ so that $\phi \circ \gamma$ is the identity on D . We shall also say that D is a retract of B .*

The following is easy.

Lemma 2.8. *Let B, D be rings and $\pi : B \twoheadrightarrow D$ has a section. If $J = \ker(\pi)$, then $\text{E}_n(B, J) = \text{E}_n(B) \cap \text{SL}_n(B, J), n \geq 3$ and $\text{ESp}_{2n}(B, J) = \text{ESp}_{2n}(B) \cap \text{Sp}_{2n}(B, J), n \geq 2$.*

It is well known that the double of a ring w.r.t. an ideal is the same as the excision algebra w.r.t. an ideal I . The following is proved in ([6], Proposition 3.1).

Proposition 2.9. *Let R be a ring of dimension d and I a finitely generated ideal of R . Consider the Cartesian square*

$$\begin{array}{ccc} C & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R/I \end{array}$$

Then, C is a finitely generated algebra of dimension d over R and integral over R . In fact, $C \simeq R \oplus I$ with the coordinate wise addition and the multiplication defined by $(a, i)(b, j) = (ab, aj + ib + ij)$, $\tilde{1} = (1, 0)$ being the identity in C . In particular, if R is an affine algebra of dimension d over a field k , then $C \simeq R \oplus I$ is also an affine algebra of dimension d over k .

Remark 2.10. We shall call C as the excision algebra of R with respect to the ideal I . There is a natural homomorphism $\mathfrak{g} : R \oplus I \longrightarrow R$ given by $(x, i) \longmapsto x + i \in R$. Clearly \mathfrak{g} has a section.

Let $\mathbf{v} = (1 + i_1, i_2, \dots, i_n) \in \text{Um}_n(R, I)$ where i_j 's are in I . Then we shall call $\tilde{\mathbf{v}} = (\tilde{1} + \tilde{i}_1, \tilde{i}_2, \dots, \tilde{i}_n) \in \text{Um}_n(R \oplus I, 0 \oplus I)$ for $\tilde{1} = (1, 0)$, $\tilde{i}_j = (0, i_j)$ to be a lift of \mathbf{v} . Note that \mathfrak{g} sends $\tilde{\mathbf{v}}$ to \mathbf{v} .

The following is from ([14], Section §5, Remarks (a)).

Lemma 2.11. (Vaserstein) For any commutative ring R , we have

$$e_1 E_{2n}(R) = e_1 \text{ESp}_{2n}(R).$$

Lemma 2.12. For any commutative ring R and an ideal I of R , we have

$$e_1 E_{2n}(R, I) = e_1 \text{ESp}_{2n}(R, I).$$

Proof. Let $R^* = R \oplus I$ and $I^* = 0 \oplus I$. Clearly the quotient map $q : R^* \longrightarrow R = R^*/I^*$ has a section. Let $\mathbf{v} \in e_1 E_{2n}(R, I)$. Then there exists $\varepsilon \in E_{2n}(R, I)$ such that $\mathbf{v} = e_1 \varepsilon$. Let $\tilde{\varepsilon} \in E_{2n}(R^*, I^*)$ be a lift of ε i.e. $\mathfrak{g}(\tilde{\varepsilon}) = \varepsilon$. Then $\tilde{\mathbf{v}} = e_1 \tilde{\varepsilon} \in \text{Um}_{2n}(R^*, I^*)$ is a lift of \mathbf{v} . Now by Lemma 2.11 we have $\tilde{\delta} \in \text{ESp}_{2n}(R^*)$ such that $\tilde{\mathbf{v}} = e_1 \tilde{\delta}$. Going modulo I^* we have $e_1 = e_1 q(\tilde{\delta})$. So replacing $\tilde{\delta}$ by $q(\tilde{\delta})^{-1} \tilde{\delta}$ if necessary we may assume that $\tilde{\delta} \in \text{ESp}_{2n}(R^*) \cap \text{Sp}_{2n}(R^*, I^*) = \text{ESp}_{2n}(R^*, I^*)$ by Lemma 2.8. So $\delta = \mathfrak{g}(\tilde{\delta}) \in \text{ESp}_{2n}(R, I)$ and $\mathbf{v} = \mathfrak{g}(\tilde{\mathbf{v}}) = e_1 \mathfrak{g}(\tilde{\delta}) = e_1 \delta \in e_1 \text{ESp}_{2n}(R, I)$. Therefore $e_1 E_{2n}(R, I) \subset e_1 \text{ESp}_{2n}(R, I)$. The reverse inclusion is obvious. \square

The following lemma follows from ([20], Theorem 2.3(e), 2.5), ([19], Proposition 2.4).

Lemma 2.13. Let the maximal spectrum $\max(R)$ of R be a disjoint union of $V(I) = \{\mathfrak{m} \in \max(R) : I \subset \mathfrak{m}\}$ and finitely many subsets V_i where each V_i , when endowed with the Zariski topology (topology induced from $\text{Spec } R$) is a noetherian space of dimension at most d . Let $(a_0, a_1, \dots, a_m) \in \text{Um}_{m+1}(R)$, $a_0 \equiv 1 \pmod{I}$ and $m \geq d+1$ and S a subset of size $d+1$ of $\{0, 1, \dots, m-1\}$. Then there exists $t_i \in I$ with $t_i = 0$ for $i \notin S$ such that $(a_0 + a_m t_0, a_1 + a_m t_1, \dots, a_{m-1} + a_m t_{m-1}) \in \text{Um}_m(R)$. In particular $\text{Um}_m(R, I) = e_1 E_m(R, I)$ for $m \geq d+2$.

Lemma 2.14. Let the maximal spectrum of R satisfy conditions as in Lemma 2.13 and $\alpha \in \text{Sp}_{2n}(R, I)$ for $2n \geq d+2$. Then $\exists \varepsilon \in \text{ESp}_{2n}(R, I)$ such that $\alpha \varepsilon = I_2 \perp \gamma$ for some $\gamma \in \text{Sp}_{2n-2}(R, I)$.

Proof. Let $\mathbf{v} = e_1 \alpha \in \text{Um}_{2n}(R, I)$. Since $2n \geq d+2$ we have $\text{Um}_{2n}(R, I) = e_1 E_{2n}(R, I) = e_1 \text{ESp}_{2n}(R, I)$ by Lemmas 2.12, 2.13. So we have an elementary symplectic matrix $\varepsilon' \in \text{ESp}_{2n}(R, I)$ such that $\mathbf{v} \varepsilon' = e_1$. Therefore the first row of the symplectic matrix

$\alpha \varepsilon'$ is e_1 and by Lemma 2.4 its second column is e_2^t . So $\alpha \varepsilon'$ will look like $\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & v_1 \\ v_2^t & 0 & \beta \end{pmatrix}$

for some $x \in I$, $v_1, v_2 \in M_{1 \times n-2}(I)$ and $\beta \in GL_{n-2}(R, I)$. Now we shall multiply $\alpha\varepsilon'$ by suitable elementary symplectic matrices of the form $se_{2j}(z)$, $j \neq 2$, $z \in I$ from the right to change all non diagonal entries of the second row to zero. Note that no entries of the first row gets affected for these multiplications. The resulting matrix is a symplectic matrix whose second row is e_2 and therefore by Lemma 2.4 its first column must be e_1^t . Thus we have an elementary symplectic matrix ε such that $\alpha\varepsilon = I_2 \perp \gamma$ for some $\gamma \in Sp_{2n-2}(R, I)$. \square

Corollary 2.15. *If $2n \geq d + 2$, then we have $\varepsilon \in ESp_{2n}(R, I)$ such that*

$$\alpha\varepsilon = \begin{cases} I_{2n-d} \perp \gamma \text{ for some } \gamma \in Sp_d(R, I) & \text{when } d \text{ is even} \\ I_{2n-d-1} \perp \gamma \text{ for some } \gamma \in Sp_{d+1}(R, I) & \text{when } d \text{ is odd.} \end{cases}$$

Definition 2.16. (Permutation matrices) *It is well known that there exists an injective group homomorphism from the permutation group S_n on n symbols to $GL_n(\mathbb{Z})$ defined by*

$$\sigma \longmapsto (e_{\sigma(1)}^t, e_{\sigma(2)}^t, \dots, e_{\sigma(n)}^t) = (e_{\sigma^{-1}(1)}, e_{\sigma^{-1}(2)}, \dots, e_{\sigma^{-1}(n)})^t.$$

If we identify each permutation with its image in $GL_n(\mathbb{Z})$ then we can view S_n as a subgroup of $GL_n(\mathbb{Z})$. Matrices in S_n are called permutation matrices.

A transposition (ij) corresponds to $E_{ji}(1)E_{ij}(-1)E_{ji}(1)\delta_j$ where δ_j is a diagonal matrix with all but (j, j) -th entry 1 and (j, j) -th entry -1 . If σ is an even permutation matrix then $\sigma \in SL_n(\mathbb{Z}) = E_n(\mathbb{Z})$ as each transposition is of determinant -1 . Thus the group of even permutation matrices is a subgroup of $E_n(\mathbb{Z})$.

Definition 2.17. (Suslin Matrix) *Given two vectors $\mathbf{v}, \mathbf{w} \in R^{r+1}$ A.A.Suslin in ([15], §5) gave an inductive process to construct the Suslin matrix $S_r(\mathbf{v}, \mathbf{w})$. We recall this process: Let $\mathbf{v} = (a_0, \mathbf{v}_1)$, $\mathbf{w} = (b_0, \mathbf{w}_1)$, where $a_0, b_0 \in R$ and $\mathbf{v}_1, \mathbf{w}_1 \in M_{1r}(R)$. Set $S_r(\mathbf{v}, \mathbf{w}) = (a_0)$ for $r = 0$ and define*

$$S_r(\mathbf{v}, \mathbf{w}) = \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(\mathbf{v}_1, \mathbf{w}_1) \\ -S_{r-1}(\mathbf{w}_1, \mathbf{v}_1)^t & b_0 I_{2^{r-1}} \end{pmatrix}$$

In ([15], Lemma 5.1) it is noted that $S_r(\mathbf{v}, \mathbf{w})S_r(\mathbf{w}, \mathbf{v})^t = \langle \mathbf{v}, \mathbf{w} \rangle I_{2^r} = S_r(\mathbf{w}, \mathbf{v})^t S_r(\mathbf{v}, \mathbf{w})$ and $\det S_r(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle^{2^{r-1}}$ for $r \geq 1$. A.A. Suslin introduced these matrices and showed that a unimodular row of the form $(a_0, a_1, a_2^2, \dots, a_r^r)$ can be completed to an invertible matrix. Following Vaserstein in [21] we shall call unimodular row of this type as factorial row and denote it by $\psi_{r!}(\mathbf{v})$ for $\mathbf{v} = (a_0, a_1, \dots, a_r)$. In fact in ([16], Proposition 2.2, Corollary 2.5) it is shown that there is an $\beta_r(\mathbf{v}, \mathbf{w})$, $\langle \mathbf{v}, \mathbf{w} \rangle = 1$ with $[\beta_r(\mathbf{v}, \mathbf{w})] = [S_r(\mathbf{v}, \mathbf{w})]$ in $K_1(R)$ whose first row is $\psi_{r!}(\mathbf{v})$. In ([11], Corollary 4.3(iii)) it is shown that if $\mathbf{v}, \mathbf{w} \in \text{Um}_{r+1}(R, I)$, $r \geq 2$ then we may assume that $\beta_r(\mathbf{v}, \mathbf{w}) \equiv I_{r+1}$ modulo I .

A.A Suslin then described a sequence of forms $J_r \in M_{2r}(R)$ by the recurrence formula:

$$J_r = \begin{cases} 1 & \text{for } r = 0, \\ J_{r-1} \perp -J_{r-1} & \text{for } r \text{ even} \\ J_{r-1} \top - J_{r-1} & \text{for } r \text{ odd} \end{cases}$$

(Here $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ while $\alpha \top \beta = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$.)

It is easy to see that $\det(J_r) = 1$ for all r , and that $J_r^t = J_r^{-1} = (-1)^{\frac{r(r+1)}{2}} J_r$. Moreover J_r is antisymmetric if $r = 4k + 1$ and $r = 4k + 2$ whereas J_r is symmetric if $r = 4k$ and $r = 4k + 3$. In ([15], Lemma 5.3) it is noted that the following formulas are valid: The Suslin identities are

$$\begin{aligned} \text{for } r = 4k : (S_r(\mathbf{v}, \mathbf{w})J_r)^t &= S_r(\mathbf{v}, \mathbf{w})J_r; \\ \text{for } r = 4k + 1 : S_r(\mathbf{v}, \mathbf{w})J_r S_r(\mathbf{v}, \mathbf{w})^t &= \langle \mathbf{v}, \mathbf{w} \rangle J_r; \\ \text{for } r = 4k + 2 : (S_r(\mathbf{v}, \mathbf{w})J_r)^t &= -S_r(\mathbf{v}, \mathbf{w})J_r; \\ \text{for } r = 4k + 3 : S_r(\mathbf{v}, \mathbf{w})J_r S_r(\mathbf{v}, \mathbf{w})^t &= \langle \mathbf{v}, \mathbf{w} \rangle J_r. \end{aligned}$$

Remark 2.18. Consider the Suslin matrix $S_r(\mathbf{v}, \mathbf{w})$, with $\langle \mathbf{v}, \mathbf{w} \rangle = 1$. By the Suslin identities $S_r(\mathbf{v}, \mathbf{w})$ is a symplectic matrix w.r.t J_r when $r \equiv 1 \pmod{4}$ and an orthogonal matrix w.r.t J_r when $r \equiv 3 \pmod{4}$. It is to be noted that J_r is merely a permutation matrix if we ignore the sign of its entries. So we shall have a permutation matrix σ_{J_r} such that $\sigma_{J_r} J_r \sigma_{J_r}^t = \psi_{2^r-1}$ when J_r is antisymmetric i.e. $r = 4k + 1, 4k + 2$. Here $\psi_n = \sum_{i=1}^n e_{2i-1, 2i} - \sum_{i=1}^n e_{2i, 2i-1}$.

When $r = 4k+1, 4k+2$, we shall denote the group of matrices $\alpha \in \text{GL}_{2n}(R)$ satisfying $\alpha J_r \alpha^t = J_r$ by $\text{Sp}_{J_r}(R)$. Clearly $\text{Sp}_{J_r}(R) = \{\alpha | \alpha J_r \alpha^t = J_r\} = \sigma_{J_r}^{-1} \text{Sp}_{2r}(R) \sigma_{J_r}$. We shall define $\text{ESp}_{J_r}(R) = \sigma_{J_r}^{-1} \text{ESp}_{2r}(R) \sigma_{J_r}$, $\text{ESp}_{J_r}(R, I) = \sigma_{J_r}^{-1} \text{ESp}_{2r}(R, I) \sigma_{J_r}$. By ([17], Corollary 1.4) and Theorem 2.6 we have $\text{ESp}_{J_r}(R, I) \subset \text{E}_{2r}(R, I)$.

The following is proved in ([11], Lemma 4.2).

Lemma 2.19. Let R be a ring and J an ideal of R . Let $\alpha \in \text{SL}_n(R)$ with $\alpha \equiv I_n \pmod{J}$. Then there exists a unique matrix $S \in \text{SL}_n(\mathbb{Z} \oplus J)$ such that $S \equiv I_n \pmod{0 \oplus J}$, and with $\mathfrak{f}(S) = \alpha$, where $\mathfrak{f} : \mathbb{Z} \oplus J \rightarrow R$ is the natural homomorphism (see Definition 2.5). Moreover, if $n = 2^m$, for some m , and α is a Suslin matrix of determinant one, then S is also a Suslin matrix of determinant one.

We recall ([2], Lemma 2.13) and include a proof for the sake of completeness.

Lemma 2.20. Let the maximal spectrum of R satisfy conditions as in Lemma 2.13, $d \geq 1$ and $S_r(\mathbf{v}, \mathbf{w}) \in \text{SL}_{2r}(R, I)$, $2^r \geq d + 2$ for $\mathbf{v}, \mathbf{w} \in \text{Um}_{r+1}(R, I)$ satisfying $\langle \mathbf{v}, \mathbf{w} \rangle = 1$. Assume that $r \equiv 1 \pmod{4}$. Then there exists $\varepsilon_{J_r} \in \text{ESp}_{J_r}(R, I)$ such that $S_r(\mathbf{v}, \mathbf{w})\varepsilon_{J_r} =$

$(I_{2r-k} \perp \gamma)\varepsilon$ for some $\gamma \in \mathrm{Sp}_k(R, I)$ and $\varepsilon \in \mathrm{E}_{2r}(R, I)$. Here $k = d + 1, d$ according as d is odd or even respectively.

Proof. Let $R^* = \mathbb{Z} \oplus I$, $I^* = 0 \oplus I$. We shall consider the corresponding Suslin matrix $S_r(\mathbf{v}^*, \mathbf{w}^*) \in \mathrm{SL}_{2r}(R^*, I^*)$ described in the Lemma 2.19. Since $r = 4k + 1$ we have $S_r(\mathbf{v}^*, \mathbf{w}^*) \in \mathrm{Sp}_{J_r}(R^*, I^*)$. So $S = \sigma_{J_r} S_r(\mathbf{v}^*, \mathbf{w}^*) \sigma_{J_r}^{-1} \in \mathrm{Sp}_{2r}(R^*, I^*)$. Therefore by Corollary 2.15 we have an $\varepsilon \in \mathrm{ESp}_{2r}(R^*, I^*)$ such that $S\varepsilon = I_{2r-k} \perp \gamma^*$ for some symplectic matrix $\gamma^* \in \mathrm{Sp}_k(R^*, I^*)$ of desired size. If we define $\varepsilon_{J_r} = \sigma_{J_r}^{-1} \varepsilon \sigma_{J_r} \in \mathrm{ESp}_{J_r}(R^*, I^*)$ then we have $S_r(R^*, I^*) \varepsilon_{J_r} = \sigma_{J_r}^{-1} (I_{2r-k} \perp \gamma) \sigma_{J_r}$.

Now after interchanging the first pair of rows and columns if necessary we may assume that $S_r(\mathbf{v}^*, \mathbf{w}^*) \varepsilon_{J_r}^* = \sigma^{-1} (I_{2r-k} \perp \gamma^*) \sigma$ for some even permutation $\sigma \in \mathrm{E}_{2r}(\mathbb{Z})$. Let $\varepsilon^* = \{ (I_{2r-k} \perp \gamma^*)^{-1} \sigma^{-1} (I_{2r-k} \perp \gamma^*) \} \sigma$. We know that $\mathrm{E}_{2r}(R^*)$ is a normal subgroup of $\mathrm{SL}_{2r}(R^*)$ ([17], Corollary 1.4). So $\varepsilon^* \in \mathrm{E}_{2r}(R^*) \cap \mathrm{SL}_{2r}(R^*, I^*)$. But R^*/I^* is a retract of R^* (Definition 2.7). Therefore $\varepsilon^* \in \mathrm{E}_{2r}(R^*, I^*)$ by Lemma 2.8 and we have $S_r(\mathbf{v}^*, \mathbf{w}^*) \varepsilon_{J_r}^* = (I_{2r-k} \perp \gamma^*) \varepsilon^*$. Taking the image in R under the natural homomorphism $\mathfrak{f} : R^* \rightarrow R$ the result follows. \square

3. EQUALITY OF ORBITS

In the previous section we have seen that both the groups $\mathrm{E}_{2n}(R, I)$ and $\mathrm{ESp}_{2n}(R, I)$ act on $\mathrm{Um}_n(R, I)$. It is natural to ask whether orbits under these two different group actions are the same. In ([4], Theorem 4.1), it is shown that this is indeed the case in the absolute case i.e. $\mathbf{v}\mathrm{E}_{2n}(R) = \mathbf{v}\mathrm{ESp}_{2n}(R)$ for $\mathbf{v} \in \mathrm{Um}_{2n}(R), n \geq 2$. In the same paper it is also proved that the result is true in the relative case too but with an extra hypothesis viz $n \geq 3$ and $2R = R$ (see Theorem 5.5). In this section we give a different proof of this result which works both in the absolute case and relative case and also in all characteristics.

Definition 3.1. Let $\mathbf{v} = (v_0, v_1, \dots, v_{2n-1}) \in R^{2n}$. Then we define $\widehat{\mathbf{v}} = \mathbf{v}\psi_{2n}$. Note that $\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle = 0$.

Lemma 3.2. ([17], Lemma 1.3) Let $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathrm{Um}_n(R)$ and $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathrm{Um}_n(R)$ be such that $\sum_{i=1}^n u_i v_i = 1$. Let $\phi : R^n \rightarrow R$ be the map sending $e_i \mapsto v_i$ where $\{e_1, e_2, \dots, e_n\}$ is the basis of R^n . Then for $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \ker(\phi)$ we have $\mathbf{w} = \sum_{i \neq j} w_i u_j (v_j e_i - v_i e_j)$.

Lemma 3.3. ([7], Lemma 1.5) Let $n \geq 2$ and $a \in I$, $\mathbf{v} \in R^{2n}$ or $a \in I$, $\mathbf{v} \in I^{2n}$. Then $1_{2n} + a\mathbf{v}^t \widehat{\mathbf{v}} \in \mathrm{ESp}_{2n}(R, I)$.

Lemma 3.4. ([7], Lemma 1.9) Let $\mathbf{w} \in R^{2n}$ and $\mathbf{v}_i \in I^{2n}, n \geq 2$ for $i = 1, 2, \dots, k$ be such that $\langle \widehat{\mathbf{v}}_i, \mathbf{w} \rangle = 0$ for all i . Then there is an element $b \in I$ such that

$$1_{2n} + \sum_{i=1}^k (\mathbf{v}_i^t \widehat{\mathbf{w}} + \mathbf{w}^t \widehat{\mathbf{v}}_i) = \prod_{i=1}^k (1_{2n} + \mathbf{v}_i^t \widehat{\mathbf{w}} + \mathbf{w}^t \widehat{\mathbf{v}}_i) \times (1_{2n} + b\mathbf{w}^t \widehat{\mathbf{w}})$$

Lemma 3.5. *Let $\alpha(X) \in \text{ESp}_{2n}(R[T, T^{-1}, X], (X))$, $n \geq 2$. Then for sufficiently large N we have $\alpha(T^N X) \in \text{ESp}_{2n}(R[T, X], (X))$.*

Proof. Let $S = R[T, T^{-1}]$. It is enough to prove the theorem for $\alpha(X) = \gamma s e_{ij}(Xf)\gamma^{-1}$, $i \neq j$, $f \in S[X]$, $\gamma \in \text{ESp}_{2n}(S[X])$. Note that if $\mathbf{v}_j = e_j \gamma^t$ i.e. \mathbf{v}_j^t is the j^{th} column of γ then $(-1)^{j+1} \widehat{\mathbf{v}}_j$ is the $\sigma(j)^{\text{th}}$ row of γ^{-1} . This is because $\gamma^t \psi_{2n} \gamma = \psi_{2n}$ gives $\widehat{\mathbf{v}}_j = e_j \gamma^t \psi_{2n} = e_j \psi_{2n} \gamma^{-1} = (-1)^{j+1} e_{\sigma(j)} \gamma^{-1}$. We prove it in the following cases.

Case - 1 $i = \sigma(j)$

In this case $\alpha(X) = \gamma s e_{i\sigma(i)}(Xf(X))\gamma^{-1} = 1_{2n} + (-1)^{i+1} Xf(X) \mathbf{v}_i^t \widehat{\mathbf{v}}_i$. We choose N sufficiently large such that $T^N \mathbf{v}_i \in M_{1,2n}(R[X, T])$ and $T^N f(X) \in R[X, T]$. Then $\alpha(T^{3N} X) \in \text{ESp}_{2n}(R[T, X], (X))$ by Lemma 3.3.

Case - 2 $i \neq \sigma(j)$

In this case $\alpha(X) = \gamma s e_{ij}(Xf(X))\gamma^{-1} = 1_{2n} + (-1)^j Xf(X) \mathbf{v}_i^t \widehat{\mathbf{v}}_{\sigma(j)} + (-1)^j Xf(X) \mathbf{v}_{\sigma(j)}^t \widehat{\mathbf{v}}_i$. Call $\mathbf{v}_i = \mathbf{v} = (v_1, v_2, \dots, v_{2n}) \in \text{Um}_{2n}(S[X])$ and $\widehat{\mathbf{v}}_{\sigma(j)} = \mathbf{w} = (w_1, w_2, \dots, w_{2n}) \in \text{Um}_{2n}(S[X])$. Note that $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ since $(-1)^j \mathbf{w}$ is the j^{th} row of γ^{-1} . Now by Lemma 3.2 $\mathbf{w} = \sum_{k \neq l} w_k u_l (v_l e_k - v_k e_l)$ where $\mathbf{u} = (u_1, u_2, \dots, u_{2n}) \in \text{Um}_{2n}(S[X])$ such that $\langle \mathbf{u}, \mathbf{v} \rangle = 1$. We have $\mathbf{v}_{\sigma(j)} = -\mathbf{w} \psi_{2n} = \sum_{k \neq l} w_k u_l ((-1)^k v_l e_{\sigma(k)} - (-1)^l v_k e_{\sigma(l)})$. Let $\mathbf{v}_{kl} = w_k u_l ((-1)^k v_l e_{\sigma(k)} - (-1)^l v_k e_{\sigma(l)})$. Then $\widehat{\mathbf{v}}_{kl} = w_k u_l (v_l e_k - v_k e_l)$ and $\langle \widehat{\mathbf{v}}_{kl}, \mathbf{v} \rangle = 0$. Note that $\mathbf{v}_{\sigma(j)} = \sum_{k \neq l} \mathbf{v}_{kl}$ and $\mathbf{w} = \widehat{\mathbf{v}}_{\sigma(j)} = \sum_{k \neq l} \widehat{\mathbf{v}}_{kl}$.

$$\begin{aligned} \alpha(X) &= 1_{2n} + (-1)^j Xf(X) \mathbf{v}_i^t \widehat{\mathbf{v}}_{\sigma(j)} + (-1)^j Xf(X) \mathbf{v}_{\sigma(j)}^t \widehat{\mathbf{v}}_i \\ &= 1_{2n} + (-1)^j Xf(X) \sum_{k \neq l} [\mathbf{v}^t \widehat{\mathbf{v}}_{kl} + \mathbf{v}_{kl}^t \widehat{\mathbf{v}}] \\ &= \prod_{k \neq l} [1_{2n} + (-1)^j Xf(X) (\mathbf{v}_{kl}^t \widehat{\mathbf{v}} + \mathbf{v}^t \widehat{\mathbf{v}}_{kl})] (1_{2n} + Xh(X) \mathbf{v}^t \widehat{\mathbf{v}}) \text{ by Lemma 3.4.} \end{aligned}$$

We choose N sufficiently large such that $T^N \mathbf{v}_{kl}, T^N \mathbf{v}$ are in $M_{1,2n}(R[X, T])$ and $T^N f(X), T^N h(X) \in R[T, X]$. Then $\alpha(T^{3N} X) \in \text{ESp}_{2n}(R[T, X], (X))$ by an argument which is implicit in ([7], Lemma 1.10). \square

Corollary 3.6. (Dilation Principle) Let $\alpha(X) \in \text{ESp}_{2n}(R_a[X], (X))$. Then for sufficiently large N we have $\beta(X) \in \text{ESp}_{2n}(R[X], (X))$ such that $\alpha(a^N X) = \beta(X)_a$.

Proof. We choose $\gamma(X, T) \in \text{ESp}_{2n}(R[X, T, 1/T], (X))$ such that $\gamma(X, a) = \alpha(X)$. Then by Lemma 3.5 we have an integer N sufficiently large such that $\gamma(T^N X, T) \in \text{ESp}_{2n}(R[X, T], (X))$. Now the result follows for $\beta(X) = \gamma(a^N X, a)$. \square

Using the dilation principle one can prove the following. The proof is similar to Vaserstein's proof of the corresponding result for elementary action (see [8], Chapter III, Section §2, Proposition 2.3, Theorem 2.4, 2.5).

Proposition 3.7. *Let $\mathbf{v}(X) \in \text{Um}_{2n}(R[X])$, $n \geq 2$ such that $\mathbf{v}(X)_s \sim_{\text{ESp}_{2n}(R_s[X])} \mathbf{v}(0)_s$. Then for sufficiently large m we have $\mathbf{v}(X + s^m Y) \sim_{\text{ESp}_{2n}(R[X, Y])} \mathbf{v}(X)$.*

Let S be a sub ring of R . Define $\mathfrak{A} = \{a \in S \mid v(X)_a \sim_{\text{ESp}_{2n}(R_a[X])} v(0)\}$ and $\mathfrak{B} = \{b \in S \mid v(X+bY) \sim_{\text{ESp}_{2n}(R[X,Y])} v(X)\}$. Then $\mathfrak{A}, \mathfrak{B}$ are ideals of S such that $\mathfrak{A} = \sqrt{\mathfrak{B}}$.

Suppose $v(X)_{\mathfrak{m}} \sim_{\text{ESp}_{2n}(R_{\mathfrak{m}}[X])} v(0)_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} of S . Then $\mathfrak{A} = \mathfrak{B} = S$. In particular $v(X+Y) \sim_{\text{ESp}_{2n}(R[X,Y])} v(X)$ and therefore $v(X) \sim_{\text{ESp}_{2n}(R[X])} v(0)$.

The following generalizes Proposition 3.7.

Proposition 3.8. Let $R = \bigoplus_{i \geq 0} R_i$ be a positively graded ring and $R_+ = \bigoplus_{i \geq 1} R_i$ the augmentation ideal of R . Let $q : R \rightarrow R_0 = R/R_+$ be the quotient map, $\mathbf{v} \in \text{Um}_{2n}(R)$ and $\mathbf{w} = q(\mathbf{v}) \in \text{Um}_{2n}(R_0)$. Suppose $\mathbf{v}_{\mathfrak{m}} \sim_{\text{ESp}_{2n}(R_{\mathfrak{m}})} \mathbf{w}_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} of R_0 . Then $\mathbf{v} \sim_{\text{ESp}_{2n}(R)} \mathbf{w}$.

Proof. Let $\phi : R[T] \rightarrow R$ be the evaluation map sending T to 1. Then ϕ has a section $\psi : R \hookrightarrow R[T]$ defined by $\psi(\sum_{i=0}^r x_i) = \sum_{i=0}^r x_i T^i$, $\deg x_i = i$ i.e. $\phi\psi = \text{id}$.

Let $\mathbf{u}(T) = \psi(\mathbf{v}) \in \text{Um}_{2n}(R[T])$. Then by our hypothesis $\mathbf{u}(T)_{\mathfrak{m}} \sim_{\text{ESp}_{2n}(R_{\mathfrak{m}}[T])} \mathbf{u}(0)_{\mathfrak{m}}$ for all maximal ideals \mathfrak{m} of R_0 . Therefore by Proposition 3.7 we have $\mathbf{u}(T) \sim_{\text{ESp}_{2n}(R[T])} \mathbf{u}(0)$. The result now follows taking the image of both sides under ϕ . \square

We shall now prove the main result of this section extending ([4], Theorem 5.5).

Theorem 3.9. Let $\mathbf{v} \in \text{Um}_{2n}(R, I)$. Then $\mathbf{v}E_{2n}(R, I) = \mathbf{v}\text{ESp}_{2n}(R, I)$.

Proof. We may assume that $n \geq 2$ as for $n = 1$ we have $E_{2n}(R, I) = \text{ESp}_{2n}(R, I)$. Clearly $\mathbf{v}\text{ESp}_{2n}(R, I) \subset \mathbf{v}E_{2n}(R, I)$. So we only need to show that $\mathbf{v}E_{2n}(R, I)$ is a subset of $\mathbf{v}\text{ESp}_{2n}(R, I)$. Let $\mathbf{w} \in \mathbf{v}E_{2n}(R, I)$. Then there exists $\alpha \in E_{2n}(R, I)$ such that $\mathbf{v}\alpha = \mathbf{w}$. Let $\alpha = \prod_{i=1}^m \gamma_i E_{s_i t_i}(x_i) \gamma_i^{-1}$, $x_i \in I$, $\gamma_i \in E_{2n}(R)$. We define $A(X_1, X_2, \dots, X_m) = \prod_{i=1}^m \gamma_i E_{s_i t_i}(X_i) \gamma_i^{-1} \in E_{2n}(R[X_1, X_2, \dots, X_m])$. Let $X = (X_1, X_2, \dots, X_m)$ denote the multi variable. Now $W(X) = \mathbf{v}A(X) \in \text{Um}_{2n}(R[X])$ such that $W(x_1, x_2, \dots, x_m) = \mathbf{w}$ and $W(\mathbf{0}) = \mathbf{v}$, $\mathbf{0} = (0, 0, \dots, 0)$.

Now for any maximal ideal \mathfrak{m} in R we have $W(X)_{\mathfrak{m}} \sim_{E_{2n}(R_{\mathfrak{m}}[X])} \mathbf{v}_{\mathfrak{m}}$ and $\mathbf{v}_{\mathfrak{m}} \sim_{E_{2n}(R_{\mathfrak{m}})} e_1$. Therefore $W(X)_{\mathfrak{m}} \in e_1 E_{2n}(R_{\mathfrak{m}}[X]) = e_1 \text{ESp}_{2n}(R_{\mathfrak{m}}[X])$ by Lemma 2.11. So we have $W(X)_{\mathfrak{m}} \sim_{\text{ESp}_{2n}(R_{\mathfrak{m}}[X])} W(\mathbf{0})_{\mathfrak{m}}$. Therefore by Proposition 3.8 we have $W(X) \sim_{\text{ESp}_{2n}(R[X])} W(\mathbf{0}) = \mathbf{v}$ i.e. there exists $B(X) \in \text{ESp}_{2n}(R[X])$ such that $W(X)B(X) = \mathbf{v}$. Replacing $B(X)$ by $B(X)B(\mathbf{0})^{-1}$ if necessary, we may assume $B(X) \in \text{ESp}_{2n}(R[X], (X))$. Now evaluating both sides at $X = (x_1, x_2, \dots, x_m)$ we have $\mathbf{w} \in \mathbf{v}\text{ESp}_{2n}(R, I)$. \square

4. IMPROVED K_1 STABILITY

Let A be a non singular affine algebra of dimension $d \geq 2$ over a perfect C_1 field k . In ([12], Theorem 1) it is shown that $\text{SL}_n(A)/E_n(A) = K_1(A)$ for $n \geq d+1$. Later in ([5], Corollary 7.7) this injective stability estimate is improved to $n = d$ when k is algebraically closed, $d \geq 3$ and $d!k = k$. In this section we shall show that these results hold true for relative K_1 with respect to a principal ideal I of A .

Theorem 4.1. Let A be an affine algebra of dimension $d \geq 2$ over a perfect C_1 field k . Then $\text{Um}_{d+1}(A, I) = e_1 \text{SL}_{d+1}(A, I)$ for any ideal I in A .

Proof. Let $\mathbf{v} \in \text{Um}_{d+1}(A, I)$ and $\tilde{\mathbf{v}} \in \text{Um}_{d+1}(A \oplus I, 0 \oplus I)$ be a lift of \mathbf{v} . By Proposition 2.9, $A \oplus I$ is also an affine algebra of dimension d over k . So by ([12], Proposition 3.1) there exists $\varepsilon \in \text{SL}_{d+1}(A \oplus I)$ such that $\tilde{\mathbf{v}}\varepsilon = e_1$. Going modulo $0 \oplus I$ we have $e_1\bar{\varepsilon} = e_1, \bar{\varepsilon} \in \text{SL}_{d+1}(A)$. Now replacing ε by $\varepsilon\bar{\varepsilon}^{-1}$ we may assume that $\varepsilon \in \text{SL}_{d+1}(A \oplus I, 0 \oplus I)$ satisfying $\tilde{\mathbf{v}}\varepsilon = e_1$. Taking projection onto A we have $\mathbf{v}\varepsilon = e_1$ for some $\varepsilon \in \text{SL}_{d+1}(A, I)$ i.e. $\mathbf{v} \in e_1\text{SL}_{d+1}(A, I)$. \square

The following result is implicit in ([5], §7).

Lemma 4.2. *Let A be a non singular affine algebra of dimension $d \geq 3$ over an algebraically closed field k , $\frac{1}{n} \in k$. Let $\mathbf{v} \in \text{Um}_d(A)$. Then \mathbf{v} can be transformed to a row of the form $(w_0, w_1, w_2, \dots, w_{d-1}^n)$ for some $\mathbf{w} = (w_0, w_1, w_2, \dots, w_{d-1}) \in \text{Um}_d(A)$ by elementary operations.*

Lemma 4.3. *Let $\mathbf{v} = (v_0, v_1, v_2, \dots, v_n) \in \text{Um}_{n+1}(R, I)$, $n \geq 2$. Then*

$$(v_0, v_1, v_2^2, \dots, v_n^n) \sim_{\text{E}_{n+1}(R, I)} (v_0, v_1, \dots, v_n^{n!}).$$

Proof. Let $(\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_n) \in \text{Um}_{n+1}(\mathbb{Z} \oplus I)$ be a lift of \mathbf{v} under \mathfrak{f} (see Definition 2.5). Then by the main theorem in introduction of [21] we have $(\tilde{v}_0, \tilde{v}_1, \tilde{v}_2^2 \dots, \tilde{v}_n^n) \sim_{\text{E}_{n+1}(\mathbb{Z} \oplus I)} (\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_n^{n!})$. But both of $(\tilde{v}_0, \tilde{v}_1, \tilde{v}_2^2 \dots, \tilde{v}_n^n), (\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_n^{n!}) \equiv e_1 \pmod{0 \oplus I}$. So by the Excision Theorem 2.6 we have $(\tilde{v}_0, \tilde{v}_1, \tilde{v}_2^2 \dots, \tilde{v}_n^n) \sim_{\text{E}_{n+1}(\mathbb{Z} \oplus I, 0 \oplus I)} (\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_n^{n!})$. Now the result follows taking the image under f in $\text{Um}_{n+1}(R, I)$. \square

Theorem 4.4. *Let A be a nonsingular affine algebra of dimension $d \geq 4$ over an algebraically closed field k , $\frac{1}{d-1!} \in k$ and I an ideal of A . Let $\mathbf{v} \in \text{Um}_d(A, I)$. Then \mathbf{v} can be transformed to a factorial row $\psi_{d-1!}(\mathbf{w}) = (w_0, w_1, w_2^2, \dots, w_{d-1}^{d-1!})$ for some $\mathbf{w} = (w_0, w_1, w_2, \dots, w_{d-1}) \in \text{Um}_d(A, I)$ by elementary operations relative to I . In particular $\text{Um}_d(A, I) = e_1\text{SL}_d(A, I)$.*

Proof. Let $\mathbf{v} = (v_0, v_1, \dots, v_{d-1}) \in \text{Um}_d(A, I)$ and $v_0 = 1 - \lambda, \lambda \in I$. By Swan's version of Bertini's Theorem ([18]) we can add suitable multiple of $\lambda v_1, \lambda v_2, \dots, \lambda v_d$ to v_0 to assume that $\bar{A} = A/(v_0)$ is a smooth affine algebra of dimension $d - 1$. So by Lemma 4.2 we have $w_1, w_2, \dots, w_{d-1} \in (\lambda)$ such that $(\bar{w}_1, \bar{w}_2, \dots, \bar{w}_{d-1}) \in \text{Um}_{d-1}(\bar{A})$ and $(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{d-1})\bar{\varepsilon} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_{d-1}^{d-1!})$ for some $\bar{\varepsilon} \in \text{E}_{d-1}(\bar{A})$. Let $\varepsilon \in \text{E}_{d-1}((\lambda)) \subset \text{E}_{d-1}(I)$ be a lift of $\bar{\varepsilon}$. Then we have $(v_1, v_2, \dots, v_{d-1})\varepsilon \equiv (w_1, w_2, \dots, w_{d-1}^{d-1!}) \pmod{Iv_0}$. So

$$\begin{aligned} (v_0, v_1, \dots, v_{d-1}) &\sim_{\text{E}_d(A, I)} (v_0, w_1, w_2, \dots, w_{d-1}^{d-1!}) \\ &\sim_{\text{E}_d(A, I)} (w_0, w_1, w_2^2, \dots, w_{d-1}^{d-1!}); v_0 = w_0, \text{ by Lemma 4.3.} \end{aligned}$$

The last assertion is clear since $\psi_{d-1!}(w)$ is the first row of $\beta(w, w') \in \text{SL}_d(A, I)$ for $w' \in \text{Um}_d(A, I)$ satisfying $\langle w, w' \rangle = 1$. \square

We don't know if the above theorem holds true for nonsingular affine algebras of dimension three. We ask the following question.

Question 4.5. Let A be a nonsingular affine algebra of dimension three over an algebraically closed field k such that $1/2 \in k$ and I an ideal of A . Then is $\text{Um}_3(A, I) = e_1\text{SL}_3(A, I)$?

The following is proved in ([22], Theorem 3.3).

Theorem 4.6. Let A be a regular ring essentially of finite type over a field k . Let $A[\underline{T}]$ denote the polynomial algebra $A[T_1, T_2, \dots, T_n]$ and (\underline{T}) the ideal (T_1, T_2, \dots, T_n) in $A[T_1, T_2, \dots, T_n]$. If $\alpha(\underline{T}) \in \text{GL}_r(A[\underline{T}], (\underline{T}))$, then $\alpha(\underline{T}) \in \text{E}_r(A[\underline{T}], (\underline{T}))$ for $r \geq 3$.

From now onwards by the term “principal ideal” we shall mean an ideal generated by a single element possibly a unit.

Theorem 4.7. Let A be an affine algebra of dimension d over a perfect C_1 field k , $d \geq 2$ and $I = (a)$ a principal ideal of A . Let $\alpha \in \text{SL}_{d+1}(A, I) \cap \text{E}(A, I)$. Then α is isotopic to identity relative to I . Moreover if A is nonsingular then,

$$\text{SL}_{d+1}(A, I) \cap \text{E}(A, I) = \text{E}_{d+1}(A, I).$$

Proof. By classical injective stability estimate we have $\alpha \in \text{E}_{d+2}(A, I)$. So we have an isotopy $\gamma(T) \in \text{E}_{d+2}(A[T], (T))$ such that $\gamma(a) = 1 \perp \alpha$ and $\gamma(0) = I_{d+2}$. Now $e_1\gamma(T) \in \text{Um}_{d+2}(A[T], (T^2 - aT))$. So by Theorem 4.1 the first row of $\gamma(T)$ is completable to a matrix $\gamma_1(T) \in \text{SL}_{d+2}(A[T], (T^2 - aT))$ i.e. $e_1\gamma(T) = e_1\gamma_1(T)$. Now note that $\gamma(T)\gamma_1(T)^{-1}$ is also an isotopy from $1 \perp \alpha$ to I_{d+1} and it will look like $\begin{pmatrix} 1 & 0 \\ * & \beta(T) \end{pmatrix}$.

Clearly $\beta(T) \in \text{SL}_{d+1}(A[T], (T))$ such that $\beta(a) = \alpha$. We define $\beta'(T) = \beta(aT)$. Then $\beta'(T) \in \text{SL}_{d+1}(A[T], I)$ is an isotopy from $\beta'(0) = I_{d+1}$ to $\beta'(1) = \alpha$ relative to $I = (a)$.

If A is nonsingular then by Theorem 4.6 we have $\beta(T) \in \text{E}_{d+1}(A[T], (T))$. So $\alpha = \beta(a) \in \text{E}_{d+1}(A, I)$. Therefore $\text{SL}_{d+1}(A, I) \cap \text{E}(A, I) \subset \text{E}_{d+1}(A, I)$. The reverse inclusion is obvious. \square

The proof of the following is verbatim that of the previous theorem. One needs to use Theorem 4.4 in place of 4.1.

Theorem 4.8. Let A be a non singular affine algebra of dimension d over an algebraically closed field k , $d \geq 3$, $d! \in k \setminus \{0\}$ and $I = (a)$ a principal ideal of A . Then

$$\text{SL}_d(A, I) \cap \text{E}(A, I) = \text{E}_d(A, I).$$

Theorems 4.7, 4.8 lead us to ask the following question.

Question 4.9. Let A be a non singular affine algebra of dimension d over an algebraically closed field k and I an arbitrary ideal of A . Then are the following true?

1. $\text{SL}_{d+1}(A, I) \cap \text{E}(A, I) = \text{E}_{d+1}(A, I)$ whenever $d \geq 2$.
2. $\text{SL}_d(A, I) \cap \text{E}(A, I) = \text{E}_d(A, I)$ whenever $d \geq 3$, $d! \in k \setminus \{0\}$.
3. $\text{SL}_{d-1}(A) \cap \text{E}(A) = \text{E}_{d-1}(A)$.

5. MAIN RESULTS

In this section we establish the main result of this article as stated in the introduction.

Lemma 5.1. *Let A be a non singular affine algebra of dimension d over an algebraically closed field k such that $\frac{1}{d!} \in k$ and $d \equiv 2 \pmod{4}$ and I a principal ideal of A . Let $\mathbf{v} \in \text{Um}_d(A, I)$. Then $\psi_{d-1!}(\mathbf{v})$ can be completed to a symplectic matrix which is congruent to identity modulo I .*

Proof. If $d = 2$ the result is obvious. So we assume that $d \geq 6$. We choose $\mathbf{w} \in \text{Um}_d(A, I)$ such that $\langle \mathbf{v}, \mathbf{w} \rangle = 1$. By Lemma 2.20 we have $\varepsilon_{J_{d-1}} \in \text{ESp}_{J_{d-1}}(A, I)$ such that $S_{d-1}(\mathbf{v}, \mathbf{w})\varepsilon_{J_{d-1}} = (I_{2^{d-1}-d} \perp \gamma)\varepsilon$ for some $\gamma \in \text{Sp}_d(A, I)$ and $\varepsilon \in \text{E}_{2^{d-1}}(A, I)$. Therefore $S_{d-1}(\mathbf{v}, \mathbf{w})$ and γ are stably elementary equivalent relative to I . Rao in ([11], Corollary 4.3 (ii)) has shown that $S_{d-1}(\mathbf{v}, \mathbf{w})$ and $\beta_{d-1}(\mathbf{v}, \mathbf{w})$ are also stably elementary equivalent relative to I . Therefore $\beta_{d-1}(\mathbf{v}, \mathbf{w})\gamma^{-1} \in \text{SL}_d(A, I) \cap \text{E}(A, I) = \text{E}_d(A, I)$ by Theorem 4.8. By Theorem 3.9 $e_1\beta_{d-1}(\mathbf{v}, \mathbf{w})\gamma^{-1} = e_1\delta$ for some $\delta \in \text{ESp}_d(A, I)$. Thus we have $\psi_{d-1!}(\mathbf{v}) = e_1\beta_{d-1}(\mathbf{v}, \mathbf{w}) = e_1\delta\gamma \in e_1\text{Sp}_d(A, I)$. \square

Theorem 5.2. *Let A be a non singular affine algebra of dimension d over an algebraically closed field k such that $d \equiv 2 \pmod{4}$, $\frac{1}{d!} \in k$ and I an ideal of A . Let $\mathbf{v} \in \text{Um}_d(A, I)$. Then \mathbf{v} can be completed to a symplectic matrix which is congruent to identity modulo I .*

Proof. We only need to consider the case $d \geq 6$. Note that if $\mathbf{v} = (1 - i_1, i_2, \dots, i_d) \in \text{Um}_d(A, I)$ then \mathbf{v} is $\text{ESp}_d(A, I)$ equivalent to $(1 - i_1, i_1i_2, i_1i_3, \dots, i_1i_d) \in \text{Um}_d(A, I)$. So without loss of generality we may assume that I is principal.

By Theorem 4.4, \mathbf{v} is elementarily equivalent to $\psi_{d-1!}(\mathbf{v}')$ relative to I for some $\mathbf{v}' \in \text{Um}_d(A, I)$. Using Theorem 3.9 we have $\mathbf{v}\varepsilon = \psi_{d-1!}(\mathbf{v})$ for some $\varepsilon \in \text{ESp}_d(A, I)$. Now the result follows by Lemma 5.1. \square

The proof of the following will follow verbatim that of ([22], Theorem 3.3) in the linear case. One may also see ([13], Theorem 3.8) for more details.

Theorem 5.3. *Let A be a regular ring essentially of finite type over a field k . Then*

$$\text{Sp}_{2r}(A[X], (X)) = \text{ESp}_{2r}(A[X], (X)), r \geq 2.$$

Theorem 5.4. *Let A be a nonsingular affine algebra of dimension d over an algebraically closed field k and $I = (a)$ a principal ideal. Assume $(d+1)! \in k^* = k \setminus \{0\}$, $d \equiv 1 \pmod{4}$. Then*

$$\text{Sp}_{d-1}(A, I) \cap \text{ESp}_{d+1}(A, I) = \text{ESp}_{d-1}(A, I).$$

Proof. Let $\sigma \in \text{Sp}_{d-1}(A, I) \cap \text{ESp}_{d+1}(A, I)$. Let $\delta(T) \in \text{ESp}_{d+1}(A[T])$ be an elementary symplectic isotopy between $\delta(a) = I_2 \perp \sigma$ and $\delta(0) = I_{d+1}$. Then $\mathbf{v}(T) = e_1\delta(T) \in \text{Um}_{d+1}(A[T], T^2 - aT)$. By previous Theorem 5.2, $\mathbf{v}(T)$ can be completed to a symplectic

matrix $\alpha(T)$ which is congruent to identity modulo $(T^2 - aT)$. So we have $e_1\delta(T) = e_1\alpha(T)$. Now $\delta(T)\alpha(T)^{-1}$ is also an isotopy between I_{d+1} and $I_2 \perp \sigma$. We have

$$\delta(T)\alpha(T)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \eta(T) \end{pmatrix}$$

for some $\eta(T) \in \text{Sp}_{d-1}(A[T], (T))$. Clearly $\eta(T)$ is a symplectic isotopy between $\eta(0) = I_{d-1}$ and $\eta(a) = \sigma$. Since A is nonsingular by Theorem 5.3 $\eta(T) \in \text{ESp}_{d-1}(A[T], (T))$. So $\sigma = \eta(a) \in \text{ESp}_{d-1}(A, I)$. Therefore $\text{Sp}_{d-1}(A, I) \cap \text{ESp}_{d+1}(A, I) \subset \text{ESp}_{d-1}(A, I)$. The reverse inclusion is trivial. \square

Theorems 5.2 and 5.4 lead us to ask the following question.

Question 5.5. *Let A be a non singular affine algebra of dimension d over an algebraically closed field k and I an ideal of A . Then are the following true in general?*

- 1 If $d! \in k \setminus \{0\}$, then $\text{Um}_d(A, I) = e_1\text{Sp}_d(A, I)$.
- 2 If $(d+1)! \in k \setminus \{0\}$, then $\text{Sp}_{d-1}(A, I) \cap \text{ESp}_{d+1}(A, I) = \text{ESp}_{d-1}(A, I)$.

6. AN EXAMPLE

Let A be an affine algebra over a field k . Let $\mathbf{v} = (v_0, v_1, \dots, v_n)$, $\mathbf{w} = (w_0, w_1, \dots, w_n) \in \text{Um}_{n+1}(A)$, $n = 2k + 1 \geq 3$ be such that $\sum_{i=0}^n v_i w_i = 1$. We can complete \mathbf{v} to an elementary matrix in $\text{E}_{n+1}(A_{v_0})$ and therefore to a matrix in $\text{ESp}_{n+1}(A_{v_0})$ by Theorem 3.9. Let $\varepsilon_1 \in \text{ESp}_{n+1}(A_{v_0})$ be such that $e_1 \varepsilon_1 = \mathbf{v}$. Since $\mathbf{v} \mathbf{w}^t = 1$, we have $\varepsilon_1 \mathbf{w}^t = (1, *, *, \dots, *)^t$. So we can suitably modify ε_1 by elementary symplectic action (multiplication by $se_{i1}(*), 1 < i$ from the left) retaining its first row and further assume that $\varepsilon_1 \mathbf{w}^t = e_1^t$.

Since v_0 is in the Jacobson radical of A_{1+v_0A} , we can also complete \mathbf{v} to an elementary matrix in $\text{E}_{n+1}(A_{1+v_0A})$. By Theorem 3.9 we shall have $\varepsilon_2 \in \text{ESp}_{n+1}(A_{1+v_0A})$ such that $e_1 \varepsilon_2 = \mathbf{v}$. After a suitable modification, if necessary, as described earlier we assume that $\varepsilon_2 \mathbf{w}^t = e_1^t$.

Now $\varepsilon_1 \varepsilon_2^{-1} \in \text{ESp}_{n+1}(A_{v_0(1+v_0A)})$ such that $e_1 \varepsilon_1 \varepsilon_2^{-1} = e_1$ and $\varepsilon_1 \varepsilon_2^{-1} e_1^t = e_1^t$. So by Lemma 2.4 $\varepsilon_1 \varepsilon_2^{-1}$ will look like $I_2 \perp \gamma$ for some $\gamma \in \text{Sp}_{n-1}(A_{v_0(1+v_0A)}) \cap \text{ESp}_{n+1}(A_{v_0(1+v_0A)})$.

If \mathbf{v} is not completable then γ must not split into two matrices over A_{v_0} and $A_{(1+v_0A)}$. Therefore γ is not contained in $\text{E}_{n-1}(A_{v_0(1+v_0A)})$. The following will follow from ([9], Claim 4, 3rd paragraph of page no 1443).

Proposition 6.1. *Let p be any prime number. Then the following assertions hold true.*

- (1) *There exists an affine domain A of dimension $p + 1$ over a C_1 field k and $\mathbf{v} = (v_0, v_1, \dots, v_p) \in \text{Um}_{p+1}(A)$ which is not completable.*
- (2) *There exists an affine domain of dimension $p + 2$ over an algebraically closed field k and a unimodular row $\mathbf{v} = (v_0, v_1, \dots, v_p) \in \text{Um}_{p+1}(A)$ which is not completable.*

We shall choose $\mathbf{v} \in \text{Um}_{p+1}(A)$ given by the first assertion and assume $n = p \geq 3$ in the above discussion. Let B be obtained from A by inverting all non zero polynomials in $k[v_0]$. Then B is an affine domain of dimension p over the C_2 field $k(v_0)$ and $A_{v_0(1+v_0A)}$ is a further localization of B . So we have $g \in B$ such that $\gamma \in \text{Sp}_{p-1}(B_g) \cap \text{ESp}_{p+1}(B_g)$. Note that B_g is also an affine domain of dimension p over the C_2 field $k(v_0)$. Also γ is not even elementary as the unimodular row is not completable. Therefore Theorem 5.4 is not true if the ground field is a C_2 field.

Now we shall choose the $\mathbf{v} \in \text{Um}_{p+1}(A)$ given by the second assertion and $n = p \geq 3, p = 3(\text{mod } 4)$ in the discussion preceding Proposition 6.1. We have $\gamma \in \text{Sp}_{p-1}(A_g) \cap \text{ESp}_{p+1}(A_g)$ but $\gamma \notin \text{ESp}_{p-1}(A_g)$ for some $g \in A$. Note that A_g is an affine domain of dimension $d = p+2, d = 1(\text{mod } 4)$. So $\gamma \in \text{Sp}_{d-3}(A_g) \cap \text{ESp}_{d-1}(A_g)$ but $\gamma \notin \text{ESp}_{d-3}(A_g)$. Therefore our estimate in Theorem 5.4 is the best possible.

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